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LETTER TO THE EDITOR

On generalisation of the Bäcklund–Calogero transformations for integrable equations

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Abstract. It is shown that the consideration of dressing (gauge) transformations non-local on all spatial variables and spectral parameters allows one to extend the class of general Bäcklund-Calogero transformations for the Kadomtsev-Petviashvili equation.

A study of the recursion and group-theoretical properties of non-linear equations integrable by the inverse spectral transform method (see, e.g., [1-4]) is an important problem of the theory of non-linear evolutions (see, e.g., [2, 3, 5]). Recently an essential step has been made in the understanding of these properties for the integrable equations in $(1+2)$ dimensions. Namely, it was shown that the usual hierarchies of integrable equations in $(1+2)$ dimensions, their symmetries and Bäcklund-Calogero transformations are generated by a single bilocal recursion operator [6-13]. The bilocality on one of the space variables (X or Y) is an essential and common feature of these results. The bilocal approach is also applicable to integrable equations in $(1+1)$ dimensions [8, 10-12].

The purpose of the present letter is to demonstrate the possibility of constructing Bäcklund-Calogero transformations (BCT) which are wider than those constructed earlier in [5-12]. These wider BCT are related to the dressing (or gauge) transformations which are non-local on all spatial variables and spectral parameters. These generalised BCT are calculated via a bilocal transformation on all spatial variables and on the spectral-parameter adjoint representation of a given spectral problem. These generalised BCT seem to include also t,x,y-dependent symmetries which were considered in [7, 14-18].

We will consider here a well known Kadomtsev-Petviashvili (KP) equation $(\sigma^2=\pm 1)$:

$$U_t(x, y, t) = U_{xxx} + 6UU_x + 3\sigma^2\partial_x^{-1}U_{yy}. \tag{1}$$

This KP equation (1) is integrable by the two-dimensional problem [1, 2]

$$L_{x,y}\psi \overset{\text{def}}{=} (\sigma\partial_y + \partial_x^2 + U(x, y, t))\psi = 0. \tag{2}$$

A change $\psi \to \hat{\psi}$ given by $\psi = \exp(\mathrm{i}\lambda x + \sigma^{-1}\lambda^2 y)\hat{\psi}(x, y, \lambda)$ $(\lambda \in \mathbb{C})$ converts (2) into the spectral problem

$$(L_{x,y} + 2\mathrm{i}\lambda\partial_x)\hat{\psi}(x, y, \lambda) = 0. \tag{3}$$

The spectral problem (3) has appeared in the framework of the $\bar{\partial}$ approach to the KP equation (see, e.g., [19]). This spectral problem is our starting point too.

Let us consider the completely non-local gauge (dressing) transformations

$$\hat{\psi}(x, y, \lambda) \to \hat{\psi}'(x', y', \lambda')$$
$$= \int d\lambda \, dx \, dy \, G(x', x; y', y; \lambda', \lambda)\hat{\psi}(x, y, \lambda) \tag{4}$$

for the problem (3) and assume that

$$(L'_{x',y'} + 2i\lambda'\partial_{x'})\hat{\psi}(x', y', \lambda') \equiv (\sigma\partial_{y'} + \partial^2_{x'} + U'(x', y') + 2i\lambda'\partial_{x'})\hat{\psi}' = 0. \tag{5}$$

As a result G obeys the equation

$$\tfrac{1}{2}i(L'_{x',y'} - L^+_{x,y})G(x', x; y', y; \lambda', \lambda) = (\lambda'\partial_{x'} + \lambda\partial_x)G \tag{6}$$

where $L^+ = -\sigma\partial_y + \partial^2_x + U(x, y)$ is the operator formally adjoint to L. Note that the bilocal quantity

$$\phi(x', x; y', y; \lambda', \lambda) \overset{\text{def}}{=} \hat{\psi}'(x', y', \lambda')\check{\psi}(x, y, \lambda)$$

where $(L^+_{x,y} - 2i\lambda\partial_x)\check{\psi} = 0$ obeys equation (6). Equation (6) is the bilocal on X, Y and λ adjoint representation of the problem (3).

An application of the completely local gauge transformations (i.e. $G = \delta(\lambda' - \lambda)\delta(y' - y)\delta(x' - x)\tilde{G}$) for the construction of the BCT in $(1+1)$ dimensions has been proposed in [20, 21]. The transformations (4) local on λ and bilocal either on X ($G = \delta(\lambda' - \lambda)\delta(y' - y)\tilde{G}$) or Y ($G = \delta(\lambda' - \lambda)\delta(x' - x)\tilde{G}$) have been used in [7–9, 12]. Infinitesimal dressing transformations (4) non-local on λ and local on X and Y ($G = \delta(x' - x)\delta(y' - y)\tilde{G}$) have beeen considered in [16].

Note that equation (6) is equivalent to

$$\tfrac{1}{2}i(L'_{x',y'} - L^+_{x,y})G = (\lambda_+\partial_+ + \lambda_-\partial_-)G \tag{7}$$

where $\lambda_\pm \overset{\text{def}}{=} \tfrac{1}{2}(\lambda' \pm \lambda)$ and $\partial_\pm \overset{\text{def}}{=} \partial_{x'} \pm \partial_x$.

The possibility of constructing different non-linear transformations associated with the KP equation (1), in particular the generalised BCT, is connected with a choice of different ansätze for G.

Here we will consider some of the simplest cases. Let us choose G as

$$G = \delta(\lambda' - \lambda) \sum_{n=0}^{N} \lambda^n_+ \varphi_n(x', x; y', y)$$

where $\varphi_0 = \delta(x' - x)\delta(y' - y)\hat{\varphi}_0$. Substituting such a G into (7) one obtains

$$\Delta(\partial_+\Lambda_+\varphi_0) = 0 \tag{8a}$$

$$\partial_+\varphi_N = 0 \qquad \Lambda_+\varphi_n = \varphi_{n-1} \qquad n = 1, \ldots, N \tag{8b}$$

where Δ is a projection operation onto the diagonal $X' = X$, $Y' = Y$:

$$\Delta Q(X', X, Y', Y) \overset{\text{def}}{=} Q|_{x'=x, y'=y}$$

and the operator Λ_+ is

$$\Lambda_+ = \tfrac{1}{2}\partial_+^{-1} i(L'_{x',y'} - L^+_{x,y})$$
$$= \tfrac{1}{2}i(\partial_{x'} + \partial_x)^{-1}[\sigma(\partial_{y'} + \partial_y) + \partial^2_{x'} - \partial^2_x + U'(x', y') - U(x, y)]. \tag{9}$$

The relations (8*b*) give $\varphi_N = f_N(x'-x, y', y)$ where f_N is an arbitrary function and $\varphi_0 = \Lambda_+^N f_N$. Substituting this expression for φ_0 into (8*a*) we finally obtain

$$\Delta(\partial_+ \Lambda_+^{N+1} f_N) = 0. \tag{10}$$

The consideration of the infinitesimal gauge transformations (4) ($\psi' = \psi + \delta\psi$, $\delta\psi = \varepsilon\psi_t$) with the same ansatz for G gives the hierarchy of the integrable equations

$$U_t(x, y, t) = \Delta(\partial_+ \Lambda_+^{N+1} \cdot 1) \tag{11}$$

and their symmetry transformations

$$\delta U = \Delta(\partial_+ \Lambda_+^{N+1} \hat{f}_N). \tag{12}$$

In formulae (11) and (12) one must put $U' \equiv U(x', y', t)$ in the operator Λ_+.

Note that $\Delta = \Delta_x \Delta_y$ where Δ_x and Δ_y are projection operations onto the diagonals $X' = X$ and $Y' = Y$ respectively:

$$\Delta_x Q(x', x; y', y) = Q|_{x'=x} \qquad \Delta_y Q(x', x; y', y) = Q|_{y'=y}.$$

The operator Λ_+ contains the derivatives $\partial_{y'}$ and ∂_y only in the combination $\partial_{y'} + \partial_y$ and hence it admits a direct projection onto the diagonal $y' = y$. As a result the BCT (10) and equations (11) can be rewritten in the form

$$\Delta_x(\partial_+ \Lambda_{x',x}^{N+1} f_N) = 0 \tag{13}$$

and

$$U_t(x, y, t) = \Delta_x(\partial_+ \Lambda_{x',x}^{N+1} \cdot 1) \tag{14}$$

where $\Lambda_{x',x} \stackrel{\text{def}}{=} \Delta_y \Lambda_+$ is the operator bilocal on X:

$$\Lambda_{x',x} = (\partial_{x'} + \partial_x)^{-1}(\sigma\partial_y + \partial_{x'}^2 - \partial_x^2 + U'(x', y) - U(x, y)).$$

The operator Λ_+ does not admit a direct projection onto $X' = X$. But one can easily check that an action of the operator $\Delta_x \Lambda_+^2$ on the vector fields of the form $\Lambda_+^m 1$ is equivalent to the action of the bilocal on the Y operator

$$L_{y',y} = -\tfrac{1}{4}\{\partial_x^2 + 2\sigma(\partial_{y'} - \partial_y) + U' + U + \partial_x^{-1}(U' + U)\partial_x$$
$$+ \partial_x^{-1}[\sigma(\partial_{y'} + \partial_y) + U' - U]\partial_x^{-1}[\sigma(\partial_{y'} + \partial_y) + U' - U]\}$$

where $U' \equiv U'(x, y')$. Correspondingly the BCT (10) and equations (11) can be represented in the forms ($N = 2M + 1$):

$$\Delta_y(\partial_x \Lambda_{y',y}^M f_M(y', y)) = 0 \tag{15}$$

and

$$U_t(x, y, t) = \Delta_y(\partial_x \Lambda_{y',y}^M \cdot 1). \tag{16}$$

The operator $\Lambda_{x',x}$ (up to the factor $\frac{1}{2}$i), the general BCT (13), the hierarchy (14) and their symmetries coincide with the bilocal on the X recursion operator, the KP hierarchy and its symmetries constructed in [8, 12] by another approach. The operator $\Lambda_{y',y}$ bilocal on Y, the BCT (15), equations (16) and their symmetries coincide with those constructed in [6, 7, 9–11].

Now let us choose G in the form

$$G=\delta(\lambda'+\lambda)\sum_{m=0}^{M}\lambda_-^m\chi_m(x',x,y',y).$$

Using this ansatz for G we obtain from (7) the following BCT:

$$\Delta(\partial_-\Lambda_-^{M+1}\xi_M)=0 \tag{17}$$

where $\xi_M=\xi_M(x'+x,y',y)$ is an arbitrary function and

$$\Lambda_-\equiv\tfrac{1}{2}\partial_-^{-1}\mathrm{i}(L'_{x',y'}-L^+_{x,y})$$
$$=\tfrac{1}{2}\mathrm{i}(\partial_{x'}-\partial_x)^{-1}[\sigma(\partial_{y'}+\partial_y)+\partial^2_{x'}-\partial^2_x+U'(x',y')-U(x,y)]. \tag{18}$$

The corresponding infinitesimal symmetry transformations are

$$\delta U=\Delta(\partial_-\Lambda^{M+1}\xi_M(x'+x,y',y)).$$

It is easy to see that $\partial_+\Lambda_+=\partial_-\Lambda_-=\frac{1}{2}\mathrm{i}(L'_{x',y'}-L^+_{x,y})$. So in fact the operator $L'_{x',y'}-L^+_{x,y}$ plays a central role in our approach.

We emphasise also that

$$\Lambda_+(x',x,y',y)\phi(\lambda,\lambda)=\lambda\phi(x',x,y',y;\lambda,\lambda)$$

and

$$\Lambda_-(x',x,y',y)\phi(\lambda,-\lambda)=\lambda\phi(x',x,y',y;\lambda,-\lambda)$$

where $\phi(x',x,y',y;\lambda',\lambda)\overset{\text{def}}{=}\hat{\psi}'(x',y',\lambda')\check{\psi}(x,y,\lambda)$.

Transformations and formulae (10)-(18) can easily be derived also for the ansätze $G=\delta(\lambda')\tilde{G}$ and $G=\delta(\lambda)\tilde{G}$. The corresponding results are given by (8)-(18) with an obvious change $\partial_+\to\partial_{x'}$, $\partial_-\to\partial_x$. In this case $f_N=f_N(x,y',y)$ and $\xi_M=\xi_M(x',y',y)$.

At last, for the ansatz

$$G=\delta(\lambda'^2-\lambda^2-4)\sum_{n=0}^{N}\lambda_+^n\varphi_n(x',x,y',y;\lambda_+)$$

relation (7) gives

$$\partial_+\varphi_N=0 \qquad \tfrac{1}{2}\mathrm{i}(L'_{x',y'}-L^+_{x,y})\varphi_0=\partial_-\varphi_1 \qquad \partial_-\varphi_0=0 \tag{19}$$

and

$$\tfrac{1}{2}\mathrm{i}(L'_{x',y'}-L^+_{x,y})\varphi_n=\partial_+\varphi_{n-1}+\partial_-\varphi_{n+1} \qquad n=1,\ldots,N-1.$$

As a result the generalised BCT are of the form

$$\Delta(P_N(\Lambda_-,\partial_1^{-1}\partial_+)\varphi_0(x'+x,y',y))=0$$

where P_N is a polynomial in Λ_- and $\partial_-^{-1}\partial_+$, the form of which is determined by recurrence relation (19).

In a similar manner one can consider also the general case

$$G=\delta(f(\lambda',\lambda))G(x',x,y',y,\lambda)$$

where $f(\lambda',\lambda)$ is some function. For example, at $f=\lambda'-\lambda^2$ and $G=\Sigma_{n=0}^N\lambda^n\varphi_n$ the generalised BCT are given by the relation

$$\Delta(\partial_xP_N(\Lambda,\partial_x^{-1}\partial_{x'})\cdot 1)=0$$

where the polynomial P_N is determined by the recurrence relation $L\varphi_n=\partial_{x'}\varphi_{n-2}+\partial_x\varphi_{n-1}$ ($\varphi_N=\varphi_{N-1}=1$) and the operator $L=\frac{1}{2}\partial_x^{-1}\mathrm{i}(L'_{x',y'}-L^+_{x,y})$.

In the one-dimensional limit, $\partial_y \to 0, \partial_{y'} \to 0$, all these formulae give the corresponding generalised transformations and symmetries for the Korteweg-de Vries equation.

One can obtain similar results for the matrix problem $(\partial_x + A\partial_y + P(x, y, t))\psi = 0$ and the problem $(\partial_x^2 - \sigma^2\partial_y^2 + \varphi(x, y)(\partial_x + \sigma\partial_y) + U(x, y))\psi = 0$ too.

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